Motivation for Gross–Siebert Program

Note Title 10/26/2019

§ Strominger–Yau–Zaslow Conjecture '96

- $X = CY^n$ admits <u>special Lagrangian</u> fibration

  $\omega^{1,1}, \ \Omega^{n,0}$ $\qquad$ $\omega|_L = 0, \ \Omega|_L = \mathrm{vol}_L$

  near <u>large complex structure limit</u>.

  $\mathrm{vol}_L \to 0$ or defined via Hodge theory

- $\check{X}$ = mirror of X is the dual torus fibration
- Complex structure of $\check{X}$ received "quantum correction" from holo. discs w/ special Lagrangian boundary condition.

$$(X, \omega) \supseteq X_0 = T^*B_0/\Lambda^\vee, \omega_0 \qquad \check{X}_0 = TB_0/\Lambda, \check{J}_0 \rightsquigarrow (\check{X}, \check{J})$$

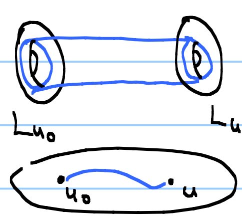
$X \downarrow B$; $X_0 \downarrow B_0$; $\check{X}_0 \downarrow B_0$

no quantum correction $\qquad$ need quantum correction

$B \longleftarrow B_0 \ \overline{\overline{}}\ B_0$ (exchange complex/symplectic affine structures)

symplectic affine $\qquad$ complex affine

$$f_i(u) = \int_{\gamma_i \in \underline{H_1(L)}} \omega \qquad \check{f}_i(u) = \int_{\check{\gamma}_i \in \underline{H_{n-1}(L)}} \mathrm{Im}\,\Omega$$



(Hitchin) affine coordinates

Complex ⟷ Symplectic affine structure of the same fibration

Legendre transform
≈
Poincare duality

In Gross–Siebert program, the integral affine manifold is constructed from the (dual) intersection complex of the <u>central fibre</u> of <u>toric degeneration</u>

Simple normal Crossing $\qquad$ LCSL

$$\left(\begin{array}{ccc} \mathrm{Fuk}^{T}(X) & \text{Homological mirror symmetry} & D^b\mathrm{Coh}(\check{X}) \\ \cup & & \cup \\ \overset{(n-p,p)}{N^*_{Z/B}} \subseteq \overset{(n,n)}{T^*B_0} & \longleftarrow \quad \overset{p}{Z} \subseteq \overset{n}{B_0} \quad \longrightarrow & \overset{(p,p)}{TZ} \subseteq \overset{(n,n)}{TB_0} \\ \text{Lagrangian} & \text{affine submanifold} & \text{hol. submanifold} \end{array}\right)$$

Hodge diamond flips

$$L = V^*/\Lambda^* \qquad \text{fibre} \qquad \check{L} = V/\Lambda = H^1(L,\mathbb{R})/H^1(L,\mathbb{Z})$$

$$= H_1(L,\mathbb{R})/H_1(L,\mathbb{Z})$$

$$= \mathrm{Hom}(\pi_1(L), U(1))$$

$$\cup$$

∇ = flat $U(1)$-connection on L

$$z_A = \exp\left(-\int_A \omega\right) \mathrm{hol}_\nabla(\partial A)$$

$$A \in H_2(X,L)$$

Notice the implicit dependence on the symplectic form

§ Large Complex Structure Limit & Tropical Geometry

What is large complex structure limit ?

Mikhalkin

$$\begin{array}{ccccc} X & \supseteq & (\mathbb{C}^*)^n & \ni & (X, Y) \\ \big\downarrow \text{toric} & & \big\downarrow \text{Log} & & \big\downarrow \\ \Delta \supseteq \Delta^\circ & = & \mathbb{R}^n & \ni & (\log|X|, \log|Y|) \end{array} \qquad \circlearrowleft H_t(X,Y) = \left(|X|^{\frac{1}{\log t}} \frac{X}{|X|},\ |Y|^{\frac{1}{\log t}} \frac{Y}{|Y|} \right)$$

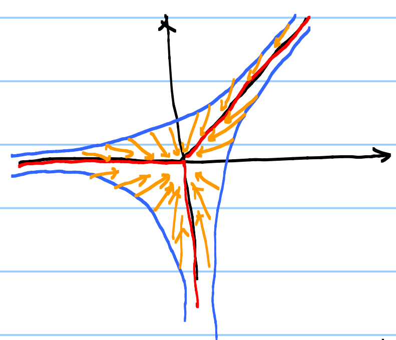
as $t \to \infty$, $\omega|_L \to 0$

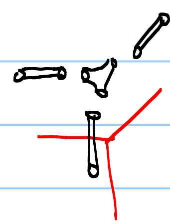
Legendre transform

$$\log|X| = x, \quad \log|Y| = y$$

What happens to holo. curves as $t \to \infty$?

ex. $\{ X + Y + 1 = 0 \} \subseteq (\mathbb{C}^*)^2$





$= V(x \oplus y \oplus 1)$

tropical curve

Moral: Large complex structure limit $\rightsquigarrow$ tropical geometry

toric degeneration

§ Landau–Ginzburg Superpotential

X^n: (toric) Fano $\xleftrightarrow{\text{mirror}}$ $(\mathbb{C}^*)^n \xrightarrow{W:\text{ holomorphic}} \mathbb{C}$ W: LG superpotential

HMS

$D^b Coh(X) \;=\!=\!=\; FS(W)$ Auroux–Kartzarkov–Orlov, Abouzaid, …

$Fuk(X) \;=\!=\!=\; D^b Sing(W)$ or $MF(W)$

$\downarrow HH^*$ $\qquad\qquad \downarrow HH^*$

$QH_s(X)$ $\qquad Jac(W) = \mathbb{C}[x_1, \cdots, x_n]/\partial_i W$

generating function of descendant GW — quantum period — $\int_\Gamma e^{-W/\hbar}\,\underline{\Omega}$ primitive form

Baranikov, Gross, …. Hong–Lin–Zhao

Q: How do we find the mirror superpotential?

(Cho–Oh '06, Hori–Vafa, Givental)

$X = X_\Delta$ toric Fano $\quad \Delta \subseteq M_\mathbb{R}$ $\qquad$ inward normal

$$\underbrace{W(L,\nabla)}_{\text{m_0 of A_∞-structure}} = \sum_{\mu(\beta)=2} n_\beta(L)\, \underbrace{\exp\left(\int_\beta \omega\right) hol_\nabla(\partial\beta)}_{} = \sum_{F:\text{ facet of } \Delta} 1\cdot z^{\nu(F)}$$

Maslov index $= 2(\beta\cdot -K_X)$ $\qquad z^{\partial\beta} = \prod_{i=1}^n z_i^{\langle \partial\beta, e_i\rangle}$, e_i basis $\in H^1(L)$

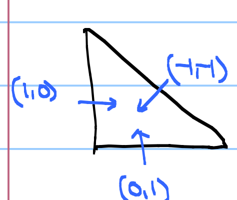
$\Downarrow$

only intersect one toric divisor

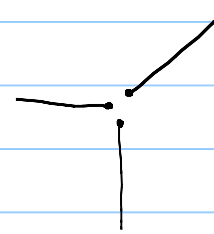
$ev_*[\mathcal{M}_1(L,\beta)]^{vir} = n_\beta(L)\,[L]$ $\qquad$ vir. dim $\mathcal{M}_0(L,\beta) = \mu(\beta)+n-3$

$n_\beta(L)$ "=" # of holo. discs in class $\beta \in H_2(X,L)$ passing through a given generic point in L.

ex. $X = \mathbb{P}^2$, $\quad W = \underline{\overset{(1,0)}{z_1} + \overset{(0,1)}{z_2}} + \dfrac{\Lambda}{z_1 z_2}$ $\;(-1,-1)$, $\quad \Lambda = \int_H \omega$



corresponding to 3 families of tropical discs



the corresponding holo. disc do NOT intersect $\{z=0\}$

$\rightsquigarrow$ fall in $\mathbb{C}^2$

$$\begin{array}{ccc} \mathbb{C}^2 & \hookleftarrow & S^1(r_1)\times S^1(r_2) \\ \downarrow & & \\ \mathbb{R}^2 & & \end{array}$$

$z_1 \rightsquigarrow D(r_1)\times \underset{\underset{S^1(r_2)}{\uparrow}}{\{z_2\}}$ β_1

$z_2 \rightsquigarrow \underset{\underset{S^1(r_1)}{\uparrow}}{\{z_1\}} \times D(r_2)$ β_2

Remark: The SYZ interpretation only give LG superpotential on a subdomain of $(\mathbb{C}^*)^n$, need renormalization

Hori-Vafa, Lin

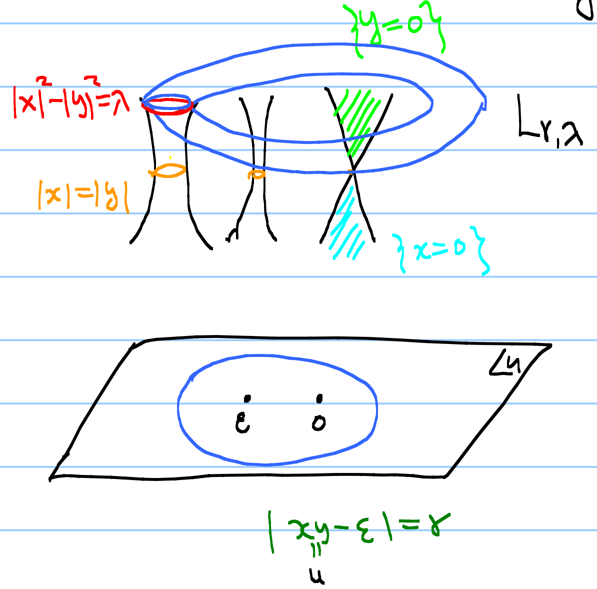
Q: What happen if there are singular fibres ?

ex. (Auroux)

$$X = \mathbb{C}^2 \setminus \{xy-\varepsilon\}, \quad \omega = \tfrac{i}{2}(dx\wedge d\bar{x} + dy\wedge d\bar{y})$$

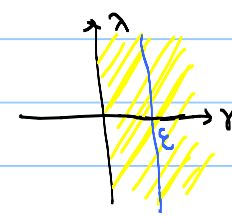
$$\Omega = \frac{dx\wedge dy}{xy-\varepsilon}$$

$$\begin{array}{ccc} X & & (x,y) \\ p\downarrow & & \downarrow \\ \mathbb{C}_u & & xy \end{array}$$



$\rightsquigarrow$

$$\begin{array}{c} X \\ \pi\downarrow \end{array} \quad \text{SYZ fibration}$$



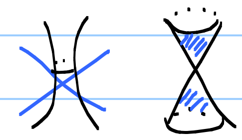
$$\left\{\begin{array}{l} r>0 \\ \lambda\in\mathbb{R} \end{array}\right\} \subseteq \mathbb{R}^2$$

$L_{r,\lambda}$ bounds a holo. disc in X

$$\begin{tikzcd} (D^2, \partial D^2) \arrow[r, "f"] \arrow[dr, "p\circ f"'] & (X, L_{r,\lambda}) \arrow[d, "p"] \\ & (\mathbb{C}\setminus\{\varepsilon\}, |u-\varepsilon| = r) \end{tikzcd}$$

maximal principle

$\rightarrow$ $p\circ f$ constant

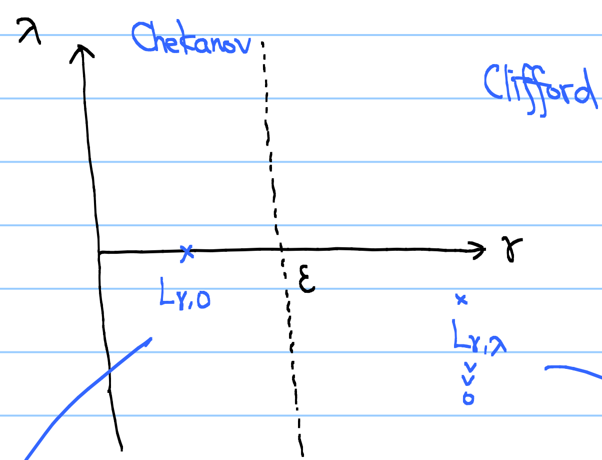
i.e $\mathrm{Im} f \subseteq$ 

$xy\neq 0$ $\quad$ $xy=0$

maximal principle

i.e $r = \varepsilon$

Q: What happens if we partial compactified X to $\mathbb{C}^2$?



$L_{r,0} = \{ |x| = |y|,\ |xy-\varepsilon| = r < \varepsilon \}$

Any Maslov index 2 disc w/ bdd on $L_{r,\lambda}$.

is a section over $\{ |xy-\varepsilon| \leq r \}$

Apply maximum principle to $\dfrac{|y|}{|x|}$

$\rightsquigarrow$ disc falls in $y = \alpha x$, $|\alpha| = 1$

only 1 family of disc β

$\rightsquigarrow$ $(u, \ )$, $\underline{W = u}$

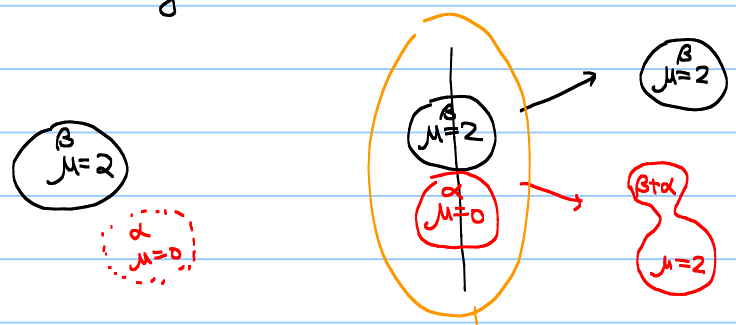
behaves like moment torus

2 families of discs β_1, β_2

$\rightsquigarrow$ (z_1, z_2), $\underline{W = z_1 + z_2}$

W jumps on the "wall"

The walls are the locus of Lagrangian fibres bounding Maslov index zero holo. discs.



$W=u \longleftrightarrow W=u(1+w)$

wall-crossing

$W=u(1+w)$ ↑ corresponds to α

$u \mapsto u(1+w)$ preserves $\frac{du}{u} \wedge \frac{dw}{w}$

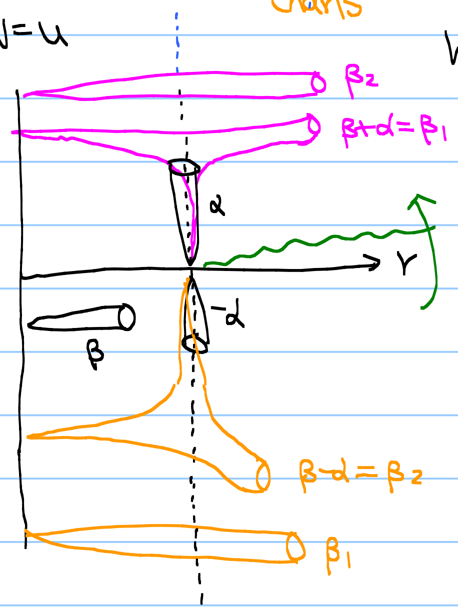
moduli space $\mathcal{M}(L, \alpha+\beta)$
has real codimension 1 boundary
so the cobordism argument has additional term

open GW is in general NOT defined.

Mirror of $= X$ $\quad \{uv = 1+w\} \subseteq \mathbb{C}^2_{uv} \times \mathbb{C}^*_w$ coincides w/ GS

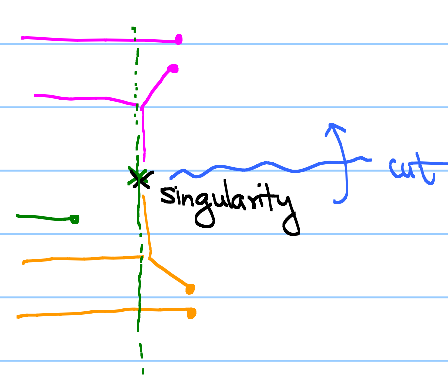
(u, w) gluing two charts $\quad (z_1, z_2) = (z_2\alpha, \alpha) \approx (v, \alpha)$, $v = z_2^{-1}$

$W=u$ $\quad W = z_1 + z_2$



branch cut
$\beta \mapsto \beta+\alpha$
monodromy

Tropical picture



The mirror constructed is generalized to all toric CY by Lau
no scattering

In general, crossing the wall induced by holo. disc α

gluing of charts : $z_\beta \longmapsto z_\beta \, h(z_\alpha)^{\langle \partial\beta, \partial\alpha \rangle}$

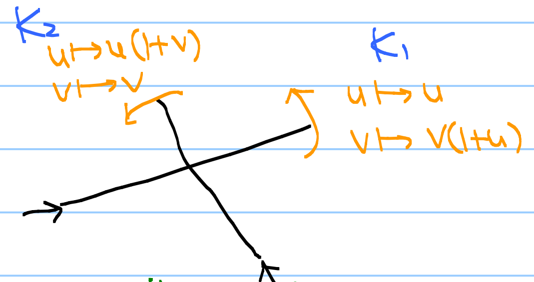
$h(z_\alpha) = 1 + O(z_\alpha)$

$\log h(z_\alpha) =$ generating function of open GW of $MI = 0$

slab function attached to the ray in the scattering diagram

Q: What happens if there are more walls / singular fibres?

ex.



$K_1 K_2 \neq K_2 K_1$

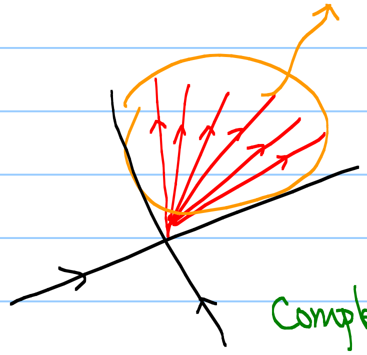
or $K_1 K_2 K_1^{-1} K_2^{-1} \neq id$

scattering diagram

(Kontsevich – Soibelman, Gross – Siebert)

$$K_1 K_2 = K_2 \boxed{} K_1$$

$\exists!$ canonical way to add rays and the attached slab functions s.t. the equality holds



Complete the scattering diagram

Remark: ① $\frac{dx}{x} \wedge \frac{dy}{y}$ is then globally defined

$\rightsquigarrow$ nowhere vanishing holo. volume form

the resulting $\check{X}$ is CY

② One can just use the initial data from the focus-focus singularity (or more general local model)

Lin

to generate the whole scattering diagram and the treatment is purely algebraic. Thus, avoid all the difficulties of

* Construction of SYZ fibration
* analysis of Floer theory
  - where J-holo. discs occurs
  - virtual fundamental class of moduli of discs.

However, don't forget the original intuition, the enumerative geometry hidden in the scattering diagram!